If the pressure at the center of the particle shown in Fig. 3.3 is denoted as p, then its average value on the two end faces that are perpendicular to the streamline are $p + \delta p_s$ and $p - \delta p_s$. Since the particle is "small," we can use a one-term Taylor series expansion for the pressure field to obtain

$$\delta p_s \approx \frac{\partial p}{\partial s}\frac{\delta s}{2}$$

Thus, if δF_{ps} is the net pressure force on the particle in the streamline direction, it follows that

$$\delta F_{ps} = (p - \delta p_s)\delta n\,\delta y - (p + \delta p_s)\delta n\,\delta y = -2\,\delta p_s\,\delta n\,\delta y$$

$$= -\frac{\partial p}{\partial s}\delta s\,\delta n\,\delta y = -\frac{\partial p}{\partial s}\delta \mathcal{V}$$

Thus, the net force acting in the streamline direction on the particle shown in Fig. 3.3 is given by

$$\sum \delta F_s = \delta \mathcal{W}_s + \delta F_{ps} = \left(-\gamma \sin\theta - \frac{\partial p}{\partial s}\right)\delta \mathcal{V} \qquad (3.3)$$

By combining Eqs. 3.2 and 3.3 we obtain the following equation of motion along the streamline direction:

$$-\gamma \sin\theta - \frac{\partial p}{\partial s} = \rho V \frac{\partial V}{\partial s} \qquad (3.4)$$

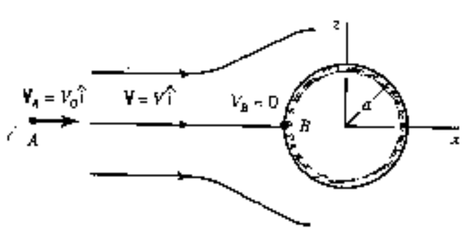
The physical interpretation of Eq. 3.4 is that a change in fluid particle speed is accomplished by the appropriate combination of pressure and particle weight along the streamline.

EXAMPLE 3.1

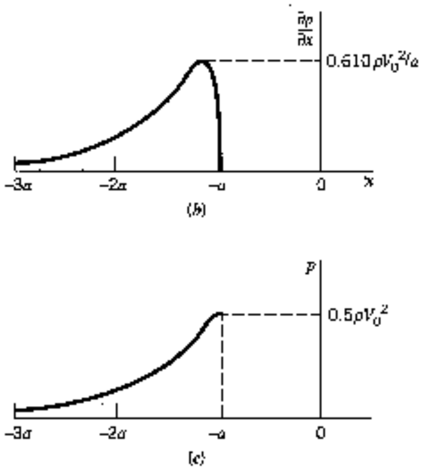
Consider the inviscid, incompressible, steady flow along the horizontal streamline A–B in front of the sphere of radius a as shown in Fig. E3.1a. From a more advanced theory of flow past a sphere, the fluid velocity along this streamline is

$$V = V_0\left(1 + \frac{a^3}{x^3}\right)$$

Determine the pressure variation along the streamline from point A far in front of the sphere ($x_A = -\infty$ and $V_A = V_0$) to point B on the sphere ($x_B = -a$ and $V_B = 0$).



(a)



■ FIGURE E3.1

SOLUTION

Since the flow is steady and inviscid, Eq. 3.4 is valid. In addition, since the streamline is horizontal, $\sin\theta = \sin 0° = 0$ and the equation of motion along the streamline reduces to

$$\frac{\partial p}{\partial s} = -\rho V \frac{\partial V}{\partial s} \tag{1}$$

With the given velocity variation along the streamline, the acceleration term is

$$V\frac{\partial V}{\partial s} = V\frac{\partial V}{\partial x} = V_0\left(1 + \frac{a^3}{x^3}\right)\left(-\frac{3V_0 a^3}{x^4}\right)$$
$$= -3V_0^2\left(1 + \frac{a^3}{x^3}\right)\frac{a^3}{x^4}$$

where we have replaced s by x since the two coordinates are identical (within an additive constant) along streamline A–B. It follows that $V\partial V/\partial s < 0$ along the streamline. The fluid slows down from V_0 far ahead of the sphere to zero velocity on the "nose" of the sphere ($x = -a$).

Thus, according to Eq. 1, to produce the given motion the pressure gradient along the streamline is

$$\frac{\partial p}{\partial x} = \frac{3\rho a^3 V_0^2 (1 + a^3/x^3)}{x^4} \tag{2}$$

This variation is indicated in Fig. E3.1*b*. It is seen that the pressure increases in the direction of flow ($\partial p/\partial x > 0$) from point A to point B. The maximum pressure gradient ($0.610\rho V_0^2/a$) occurs just slightly ahead of the sphere ($x = -1.205a$). It is the pressure gradient that slows the fluid down from $V_A = V_0$ to $V_B = 0$.

The pressure distribution along the streamline can be obtained by integrating Eq. 2 from $p = 0$ (gage) at $x = -\infty$ to pressure p at location x. The result, plotted in Fig. E3.1*c*, is

$$p = -\rho V_0^2\left[\left(\frac{a}{x}\right)^3 + \frac{(a/x)^6}{2}\right] \qquad \textbf{(Ans)}$$

The pressure at B, a stagnation point since $V_B = 0$, is the highest pressure along the streamline ($p_B = \rho V_0^2/2$). As shown in Chapter 9, this excess pressure on the front of the sphere (i.e., $p_B > 0$) contributes to the net drag force on the sphere. Note that the pressure gradient and pressure are directly proportional to the density of the fluid, a representation of the fact that the fluid inertia is proportional to its mass.

Equation 3.4 can be rearranged and integrated as follows. First, we note from Fig. 3.3 that along the streamline $\sin\theta = dz/ds$. Also, we can write $V dV/ds = \frac{1}{2} d(V^2)/ds$. Finally, along the streamline $\partial p/\partial s = dp/ds$. These ideas combined with Eq. 3.4 give the following result valid along a streamline

$$-\gamma \frac{dz}{ds} - \frac{dp}{ds} = \frac{1}{2}\rho \frac{d(V^2)}{ds}$$

This simplifies to

$$dp + \frac{1}{2}\rho d(V^2) + \gamma\, dz = 0 \qquad \text{(along a streamline)} \tag{3.5}$$

which, for constant density, can be integrated to give

$$p + \tfrac{1}{2}\rho V^2 + \gamma z = \text{constant along streamline} \tag{3.6}$$

V3.1 Balancing ball

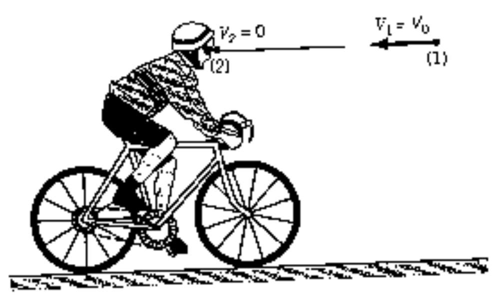
This is the celebrated *Bernoulli Equation*—a very powerful tool in fluid mechanics.

EXAMPLE 3.2

Consider the flow of air around a bicyclist moving through still air with velocity V_0 as is shown in Fig. E3.2. Determine the difference in the pressure between points (1) and (2).

SOLUTION

In a coordinate system fixed to the bike, it appears as though the air is flowing steadily toward the bicyclist with speed V_0. If the assumptions of Bernoulli's equation are valid (steady,



■ FIGURE E3.2

incompressible, inviscid flow), Eq. 3.6 can be applied as follows along the streamline that passes through (1) and (2).

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

We consider (1) to be in the free stream so that $V_1 = V_0$ and (2) to be at the tip of the bicyclist's nose and assume that $z_1 = z_2$ and $V_2 = 0$ (both of which, as is discussed in Section 3.5, are reasonable assumptions). It follows that the pressure of (2) is greater than that at (1) by an amount

$$p_2 - p_1 = \tfrac{1}{2}\rho V_1^2 = \tfrac{1}{2}\rho V_0^2 \qquad \text{(Ans)}$$

A similar result was obtained in Example 3.1 by integrating the pressure gradient, which was known because the velocity distribution along the streamline, $V(s)$, was known. The Bernoulli equation is a general integration of **F** = *m***a**. To determine $p_2 - p_1$, knowledge of the detailed velocity distribution is not needed—only the "boundary conditions" at (1) and (2) are required. Of course, knowledge of the value of V along the streamline is needed to determine the pressure at points between (1) and (2). Note that if we measure $p_2 - p_1$ we can determine the speed, V_0. As discussed in Section 3.5, this is the principle upon which many velocity measuring devices are based.

If the bicyclist were accelerating or decelerating, the flow would be unsteady (i.e., $V_0 \neq$ constant) and the above analysis would be incorrect since Eq. 3.6 is restricted to steady flow.

3.3 F = *ma* Normal to a Streamline

We again consider the force balance on the fluid particle shown in Fig. 3.3. This time, however, we consider components in the normal direction, $\hat{\mathbf{n}}$, and write Newton's second law in this direction as

$$\sum \delta F_n = \frac{\delta m\, V^2}{\Re} = \frac{\rho\, \delta \forall\, V^2}{\Re} \qquad (3.7)$$

where $\sum \delta F_n$ represents the sum of n components of all the forces acting on the particle. We assume the flow is steady with a normal acceleration $a_n = V^2/\Re$, where $\Re$ is the local radius of curvature of the streamlines.

We again assume that the only forces of importance are pressure and gravity. Using the method of Section 3.2 for determining forces along the streamline, the net force acting in the normal direction on the particle shown in Fig. 3.3 is determined to be

$$\sum \delta F_n = \delta \mathcal{W}_n + \delta F_{pn} = \left(-\gamma \cos\theta - \frac{\partial p}{\partial n}\right)\delta\forall \qquad (3.8)$$

where $\partial p/\partial n$ is the pressure gradient normal to the streamline. By combining Eqs. 3.7 and 3.8 and using the fact that along a line normal to the streamline $\cos\theta = dz/dn$ (see Fig. 3.3), we obtain the following equation of motion along the normal direction

$$-\gamma \frac{dz}{dn} - \frac{\partial p}{\partial n} = \frac{\rho V^2}{\Re} \qquad (3.9)$$

The physical interpretation of Eq. 3.9 is that a change in the direction of flow of a fluid particle (i.e., a curved path, $\mathcal{R} < \infty$) is accomplished by the appropriate combination of pressure gradient and particle weight normal to the streamline. By integration of Eq. 3.9, the final form of Newton's second law applied across the streamlines for steady, inviscid, incompressible flow is obtained as

$$p + \rho \int \frac{V^2}{\mathcal{R}}\, dn + \gamma z = \text{constant across the streamline} \qquad (3.10)$$

EXAMPLE 3.3

Shown in Figs. E3.3*a*,*b* are two flow fields with circular streamlines. The velocity distributions are

$$V(r) = C_1 r \qquad \text{for case } (a)$$

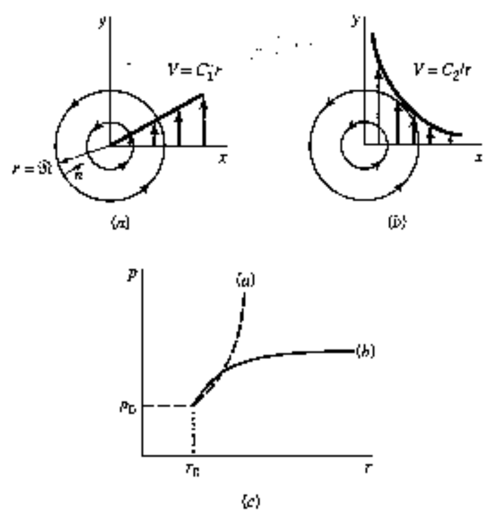
and

$$V(r) = \frac{C_2}{r} \qquad \text{for case } (b)$$

where C_1 and C_2 are constant. Determine the pressure distributions, $p = p(r)$, for each, given that $p = p_0$ at $r = r_0$.

SOLUTION

We assume the flows are steady, inviscid, and incompressible with streamlines in the horizontal plane ($dz/dn = 0$). Since the streamlines are circles, the coordinate n points in a



■ FIGURE E3.3

EXAMPLE 3.6

An airplane flies 100 mi/hr at an elevation of 10,000 ft in a standard atmosphere as shown in Fig. E3.6. Determine the pressure at point (1) far ahead of the airplane, the pressure at the stagnation point on the nose of the airplane, point (2), and the pressure difference indicated by a Pitot-static probe attached to the fuselage.

SOLUTION

From Table C.1 we find that the static pressure at the altitude given is

$$p_1 = 1456 \text{ lb/ft}^2 \text{ (abs)} = 10.11 \text{ psia} \qquad \textbf{(Ans)}$$

Also, the density is $\rho = 0.001756 \text{ slug/ft}^3$.

If the flow is steady, inviscid, and incompressible and elevation changes are neglected, Eq. 3.6 becomes

$$p_2 = p_1 + \frac{\rho V_1^2}{2}$$

With $V_1 = 100$ mi/hr $= 146.7$ ft/s and $V_2 = 0$ (since the coordinate system is fixed to the airplane) we obtain

$$\begin{aligned} p_2 &= 1456 \text{ lb/ft}^2 + (0.001756 \text{ slugs/ft}^3)(146.7 \text{ ft/s})^2/2 \\ &= (1456 + 18.9) \text{ lb/ft}^2 \text{ (abs)} \end{aligned}$$

Hence, in terms of gage pressure

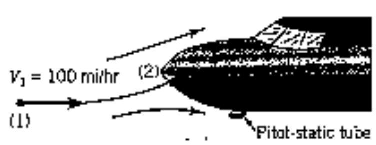
$$p_2 = 18.9 \text{ lb/ft}^2 = 0.1313 \text{ psi} \qquad \textbf{(Ans)}$$

Thus, the pressure difference indicated by the Pitot-static tube is

$$p_2 - p_1 = \frac{\rho V_1^2}{2} = 0.1313 \text{ psi} \qquad \textbf{(Ans)}$$

Note that it is very easy to obtain incorrect results by using improper units. Do not add lb/in.^{2} and lb/ft^2. Note that $(\text{slug/ft}^3)(\text{ft}^2/\text{s}^2) = (\text{slug}\cdot\text{ft/s}^2)/(\text{ft}^2) = \text{lb/ft}^2$.

It was assumed that the flow is incompressible—the density remains constant from (1) to (2). However, since $\rho = p/RT$, a change in pressure (or temperature) will cause a change in density. For this relatively low speed, the ratio of the absolute pressures is nearly unity [i.e., $p_1/p_2 = (10.11 \text{ psia})/(10.11 + 0.1313 \text{ psia}) = 0.987$], so that the density change is negligible. However, at high speed it is necessary to use compressible flow concepts to obtain accurate results.



■ FIGURE E3.6

direction opposite of that of the radial coordinate, $\partial/\partial n = -\partial/\partial r$, and the radius of curvature is given by $\mathcal{R} = r$. Hence, Eq. 3.9 becomes

$$\frac{\partial p}{\partial r} = \frac{\rho V^2}{r}$$

For case (a) this gives

$$\frac{\partial p}{\partial r} = \rho C_1^2 r$$

while for case (b) it gives

$$\frac{\partial p}{\partial r} = \frac{\rho C_2^2}{r^3}$$

For either case the pressure increases as r increases since $\partial p/\partial r > 0$. Integration of these equations with respect to r, starting with a known pressure $p = p_0$ at $r = r_0$, gives

$$p = \frac{1}{2}\rho C_1^2(r^2 - r_0^2) + p_0 \qquad \text{(Ans)}$$

for case (a) and

$$p = \frac{1}{2}\rho C_2^2\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + p_0 \qquad \text{(Ans)}$$

for case (b). These pressure distributions are sketched in Fig. E3.3c. The pressure distributions needed to balance the centrifugal accelerations in cases (a) and (b) are not the same because the velocity distributions are different. In fact for case (a) the pressure increases without bound as $r \rightarrow \infty$, while for case (b) the pressure approaches a finite value as $r \rightarrow \infty$. The streamline patterns are the same for each case, however.

Physically, case (a) represents rigid body rotation (as obtained in a can of water on a turntable after it has been "spun up") and case (b) represents a free vortex (an approximation to a tornado or the swirl of water in a drain, the "bathtub vortex").

3.4 Physical Interpretation

An alternate but equivalent form of the Bernoulli equation is obtained by dividing each term of Eq. 3.6 by the specific weight, γ, to obtain

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant on a streamline} \qquad (3.11)$$

Each of the terms in this equation has the units of length and represents a certain type of head.

The elevation term, z, is related to the potential energy of the particle and is called the *elevation head*. The pressure term p/γ, is called the *pressure head* and represents the height of a column of the fluid that is needed to produce the pressure p. The velocity term, $V^2/2g$, is the *velocity head* and represents the vertical distance needed for the fluid to fall freely (neglecting friction) if it is to reach velocity V from rest. The Bernoulli equation states that the sum of the pressure head, the velocity head, and the elevation head is constant along a streamline.

SOLUTION

For steady, inviscid, incompressible flow the Bernoulli equation applied between points
and (2) is

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

With the assumptions that $p_1 = p_2 = 0$, $z_1 = h$, and $z_2 = 0$, Eq. 1 becomes

$$\tfrac{1}{2}V_1^2 + gh = \tfrac{1}{2}V_2^2$$

Although the water level remains constant (h = constant), there is an average velocity,
across section (1) because of the flow from the tank. From Eq. 3.16 for steady incompr
ible flow, conservation of mass requires $Q_1 = Q_2$, where $Q = AV$. Thus, $A_1V_1 = A_2V_2$

$$\frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2$$

Hence,

$$V_1 = \left(\frac{d}{D}\right)^2 V_2$$

Equations 1 and 3 can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1 - (d/D)^4}}$$

Thus, with the given data

$$V_2 = \sqrt{\frac{2(9.81\ \text{m/s}^2)(2.0\ \text{m})}{1 - (0.1\ \text{m}/1\ \text{m})^4}} = 6.26\ \text{m/s}$$

and

$$Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.1\ \text{m})^2(6.26\ \text{m/s}) = 0.0492\ \text{m}^3/\text{s} \qquad \text{(A}$$

In this example we have not neglected the kinetic energy of the water in the t
($V_1 \neq 0$). If the tank diameter is large compared to the jet diameter ($D \gg d$), Eq. 3 in
cates that $V_1 \ll V_2$ and the assumption that $V_1 \approx 0$ would be reasonable. The error ass
ated with this assumption can be seen by calculating the ratio of the flowrate assuming $V_1 \neq$
denoted Q, to that assuming $V_1 = 0$, denoted Q_0. This ratio, written as

$$\frac{Q}{Q_0} = \frac{V_2}{V_2|_{D=\infty}} = \frac{\sqrt{2gh/[1 - (d/D)^4]}}{\sqrt{2gh}} = \frac{1}{\sqrt{1 - (d/D)^4}}$$

is plotted in Fig. E3.7*b*. With $0 < d/D < 0.4$ it follows that $1 < Q/Q_0 \leq 1.01$, and the e
ror in assuming $V_1 = 0$ is less than 1%. Thus, it is often reasonable to assume $V_1 = 0$.

The fact that a kinetic energy change is often accompanied by a change in pressure is
shown by Example 3.8.

The *mass flowrate* from an outlet, $\dot{m}$ (slugs/s or kg/s), is given by $\dot{m} = \rho Q$, where Q (ft^3/s or m^3/s) is the *volume flowrate*. If the outlet area is A and the fluid flows across this area (normal to the area) with an average velocity V, then the volume of the fluid crossing this area in a time interval δt is $VA\,\delta t$, equal to that in a volume of length $V\,\delta t$ and cross-sectional area A (see Fig. 3.9). Hence, the volume flowrate (volume per unit time) is $Q = VA$. Thus, $\dot{m} = \rho VA$. To conserve mass, the inflow rate must equal the outflow rate. If the inlet is designated as (1) and the outlet as (2), it follows that $\dot{m}_1 = \dot{m}_2$. Thus, conservation of mass requires

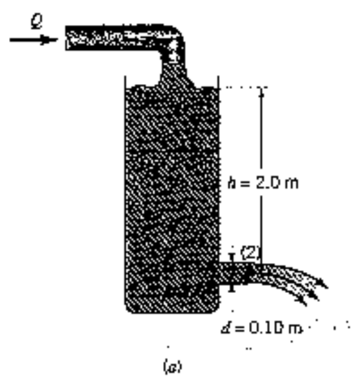
$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

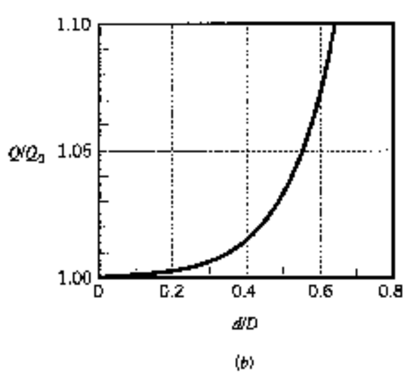
If the density remains constant, then $\rho_1 = \rho_2$ and the above becomes the *continuity equation* for incompressible flow

$$A_1 V_1 = A_2 V_2, \text{ or } Q_1 = Q_2 \tag{3.16}$$

EXAMPLE 3.7

A stream of water of diameter $d = 0.1$ m flows steadily from a tank of diameter $D = 1.0$ m as shown in Fig. E3.7*a*. Determine the flow rate, Q, needed from the inflow pipe if the water depth remains constant, $h = 2.0$ m.





■ FIGURE E3.7

Note that the value of V_3 is determined strictly by the value of p_1 (and the assumptions involved in the Bernoulli equation), independent of the "shape" of the nozzle. The pressure head within the tank, $p_1/\gamma = (3.0 \text{ kPa})/(9.81 \text{ m/s}^2)(1.26 \text{ kg/m}^3) = 243 \text{ m}$, is converted to the velocity head at the exit, $V_2^2/2g = (69.0 \text{ m/s})^2/(2 \times 9.81 \text{ m/s}^2) = 243 \text{ m}$. Although we used gage pressure in the Bernoulli equation ($p_3 = 0$), we had to use absolute pressure in the perfect gas law when calculating the density.

The pressure within the hose can be obtained from Eq. 1 and the continuity equation (Eq. 3.16)

$$A_2 V_2 = A_3 V_3$$

Hence,

$$V_2 = A_3 V_3 / A_2 = \left(\frac{d}{D}\right)^2 V_3 = \left(\frac{0.01 \text{ m}}{0.03 \text{ m}}\right)^2 (69.0 \text{ m/s})$$
$$= 7.67 \text{ m/s}$$

and from Eq. 1

$$p_2 = 3.0 \times 10^3 \text{ N/m}^2 - \tfrac{1}{2}(1.26 \text{ kg/m}^3)(7.67 \text{ m/s})^2$$
$$= (3000 - 37.1)\text{N/m}^2 = 2963 \text{ N/m}^2 \qquad \textbf{(Ans)}$$

In the absence of viscous effects the pressure throughout the hose is constant and equal to p_2. Physically, the decreases in pressure from p_1 to p_2 to p_3 accelerate the air and increase its kinetic energy from zero in the tank to an intermediate value in the hose and finally to its maximum value at the nozzle exit. Since the air velocity in the nozzle exit is nine times that in the hose, most of the pressure drop occurs across the nozzle ($p_1 = 3000 \text{ N/m}^2$, $p_2 = 2963 \text{ N/m}^2$ and $p_3 = 0$).

Since the pressure change from (1) to (3) is not too great [that is, in terms of absolute pressure $(p_1 - p_3)/p_1 = 3.0/101 = 0.03$], it follows from the perfect gas law that the density change is also not significant. Hence, the incompressibility assumption is reasonable for this problem. If the tank pressure were considerably larger or if viscous effects were important, the above results would be incorrect.

In many situations the combined effects of kinetic energy, pressure, and gravity are important. Example 3.9 illustrates this.

EXAMPLE 3.9

Water flows through a pipe reducer as is shown in Fig. E3.9. The static pressures at (1) and (2) are measured by the inverted U-tube manometer containing oil of specific gravity, SG, less than one. Determine the manometer reading, h.

SOLUTION

With the assumptions of steady, inviscid, incompressible flow, the Bernoulli equation can be written as

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

EXAMPLE 3.8

Air flows steadily from a tank, through a hose of diameter $D = 0.03\,\text{m}$ and exits to the atmosphere from a nozzle of diameter $d = 0.01\,\text{m}$ as shown in Fig. E3.8. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure. Determine the flowrate and the pressure in the hose.

SOLUTION

If the flow is assumed steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline shown as

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3$$

With the assumption that $z_1 = z_2 = z_3$ (horizontal hose), $V_1 = 0$ (large tank), and $p_3 = 0$ (free jet) this becomes

$$V_3 = \sqrt{\frac{2p_1}{\rho}}$$

and

$$p_2 = p_1 - \tfrac{1}{2}\rho V_2^2 \qquad \textbf{(1)}$$

The density of the air in the tank is obtained from the perfect gas law, using standard absolute pressure and temperature, as

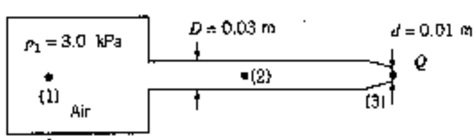
$$\rho = \frac{p_1}{RT_1}$$
$$= [(3.0 + 101)\,\text{kN/m}^2] \times \frac{10^3\,\text{N/kN}}{(286.9\,\text{N}\cdot\text{m/kg}\cdot\text{K})(15 + 273)\text{K}}$$
$$= 1.26\,\text{kg/m}^3$$

Thus, we find that

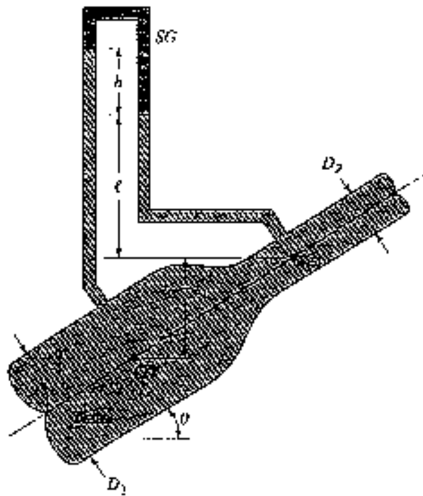
$$V_3 = \sqrt{\frac{2(3.0 \times 10^3\,\text{N/m}^2)}{1.26\,\text{kg/m}^3}} = 69.0\,\text{m/s}$$

or

$$Q = A_3V_3 = \frac{\pi}{4}d^2V_3 = \frac{\pi}{4}(0.01\,\text{m})^2(69.0\,\text{m/s})$$
$$= 0.00542\,\text{m}^3/\text{s} \qquad \textbf{(Ans)}$$



■ FIGURE E3.8



■ FIGURE E3.9

The continuity equation (Eq. 3.16) provides a second relationship between V_1 and V_2 if we assume the velocity profiles are uniform at those two locations and the fluid incompressible:

$$Q = A_1V_1 = A_2V_2$$

By combining these two equations we obtain

$$p_1 - p_2 = \gamma(z_2 - z_1) + \tfrac{1}{2}\rho V_2^2[1 - (A_2/A_1)^2] \tag{1}$$

This pressure difference is measured by the manometer and can be determined by using the pressure-depth ideas developed in Chapter 2. Thus,

$$p_1 - \gamma(z_2 - z_1) - \gamma\ell - \gamma h + SG\,\gamma h + \gamma\ell = p_2$$

or

$$p_1 - p_2 = \gamma(z_2 - z_1) + (1 - SG)\gamma h \tag{2}$$

As discussed in Chapter 2, this pressure difference is neither merely γh nor $\gamma(h + z_1 - z_2)$.

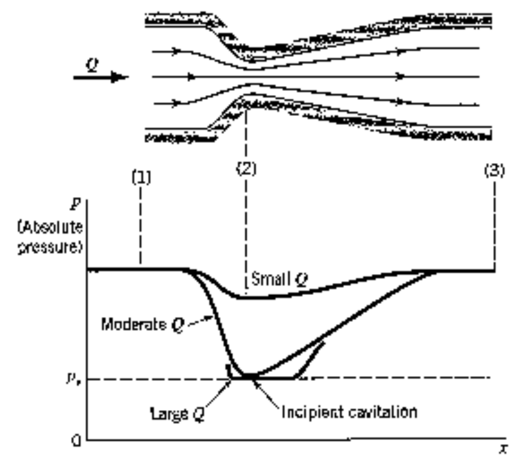
Equations 1 and 2 can be combined to give the desired result as follows

$$(1 - SG)\gamma h = \frac{1}{2}\rho V_2^2\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

or since $V_2 = Q/A_2$

$$h = (Q/A_2)^2\frac{1 - (A_2/A_1)^2}{2g(1 - SG)} \tag{Ans}$$

The difference in elevation, $z_1 - z_2$, was not needed because the change in elevation term in the Bernoulli equation exactly cancels the elevation term in the manometer equation. However, the pressure difference, $p_1 - p_2$, depends on the angle θ, because of the elevation, $z_1 - z_2$, in Eq. 1. Thus, for a given flowrate, the pressure difference, $p_1 - p_2$, as measured by a pressure gage would vary with θ, but the manometer reading, h, would be independent of θ.



■ **FIGURE 3.10 Pressure variation and cavitation in a variable area pipe.**

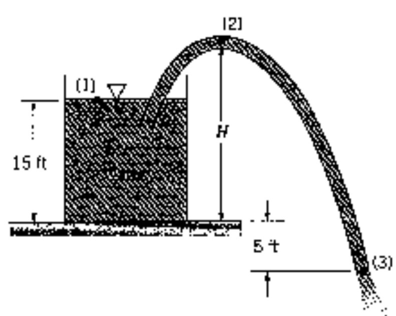
In general, an increase in velocity is accompanied by a decrease in pressure. If the differences in velocity are considerable, the differences in pressure can also be considerable. For flows of liquids, this may result in cavitation, a potentially dangerous situation that results when the liquid pressure is reduced to the vapor pressure and the liquid "boils."

V3.6 Venturi channel

One way to produce cavitation in a flowing liquid is noted from the Bernoulli equation. If the fluid velocity is increased (for example, by a reduction in flow area as shown in Fig. 3.10) the pressure will decrease. This pressure decrease (needed to accelerate the fluid through the constriction) can be large enough so that the pressure in the liquid is reduced to its vapor pressure.

EXAMPLE 3.10

Water at 60°F is siphoned from a large tank through a constant diameter hose as shown in Fig. E3.10. Determine the maximum height of the hill, H, over which the water can be siphoned without cavitation occurring. The end of the siphon is 5 ft below the bottom of the tank. Atmospheric pressure is 14.7 psia.



■ **FIGURE E3.10**

SOLUTION

If the flow is steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) as follows

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2 = p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3 \tag{1}$$

With the tank bottom as the datum, we have $z_1 = 15\text{ ft}$, $z_2 = H$, and $z_3 = -5\text{ ft}$. Also, $V_1 = 0$ (large tank), $p_1 = 0$ (open tank), $p_3 = 0$ (free jet), and from the continuity equation $A_2V_2 = A_3V_3$, or because the hose is constant diameter, $V_2 = V_3$. Thus, the speed of the fluid in the hose is determined from Eq. 1 to be

$$V_3 = \sqrt{2g(z_1 - z_3)} = \sqrt{2(32.2 \text{ ft/s}^2)[15 - (-5)] \text{ ft}}$$
$$= 35.9 \text{ ft/s} = V_2$$

Use of Eq. 1 between points (1) and (2) then gives the pressure p_2 at the top of the hill as

$$p_2 = p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 - \tfrac{1}{2}\rho V_2^2 - \gamma z_2 = \gamma(z_1 - z_2) - \tfrac{1}{2}\rho V_2^2 \tag{2}$$

From Table B.1, the vapor pressure of water at 60°F is 0.256 psia. Hence, for incipient cavitation the lowest pressure in the system will be $p = 0.256$ psia. Careful consideration of Eq. 2 and Fig. E3.10 will show that this lowest pressure will occur at the top of the hill. Since we have used gage pressure at point (1) ($p_1 = 0$), we must use gage pressure at point (2) also. Thus, $p_2 = 0.256 - 14.7 = -14.4$ psi and Eq. 2 gives

$$(-14.4 \text{ lb/in.}^2)(144 \text{ in.}^2/\text{ft}^2) = (62.4 \text{ lb/ft}^3)(15 - H)\text{ft} - \tfrac{1}{2}(1.94 \text{ slugs/ft}^3)(35.9 \text{ ft/s})^2$$

or

$$H = 28.2 \text{ ft} \qquad \textbf{(Ans)}$$

For larger values of H, vapor bubbles will form at point (2) and the siphon action may stop.

Note that we could have used absolute pressure throughout ($p_2 = 0.256$ psia and $p_1 = 14.7$ psia) and obtained the same result. The lower the elevation of point (3), the larger the flowrate and, therefore, the smaller the value of H allowed.

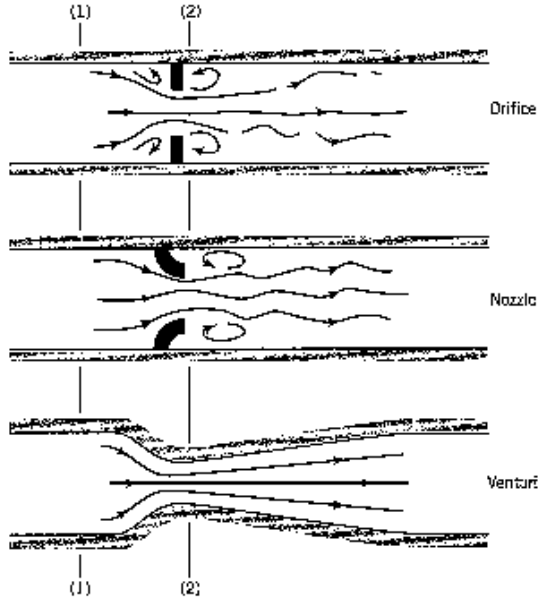
We could also have used the Bernoulli equation between (2) and (3), with $V_2 = V_3$, to obtain the same value of H. In this case it would not have been necessary to determine V_2 by use of the Bernoulli equation between (1) and (3).

The above results are independent of the diameter and length of the hose (provided viscous effects are not important). Proper design of the hose (or pipe) is needed to ensure that it will not collapse due to the large pressure difference (vacuum) between the inside and the outside of the hose.

3.6.3 Flowrate Measurement

Many types of devices using principles involved in the Bernoulli equation have been developed to measure fluid velocities and flowrates.

An effective way to measure the flowrate through a pipe is to place some type of restriction within the pipe as shown in Fig. 3.11 and to measure the pressure difference between the low-velocity, high-pressure upstream section (1), and the high-velocity, low-pressure



■ **FIGURE 3.11** **Typical devices for measuring flowrate in pipes.**

downstream section (2). Three commonly used types of flowmeters are illustrated: the *orifice meter,* the *nozzle meter,* and the *Venturi meter.* The operation of each is based on the same physical principles—an increase in velocity causes a decrease in pressure.

We assume the flow is horizontal ($z_1 = z_2$), steady, inviscid, and incompressible between points (1) and (2). The Bernoulli equation becomes

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_2 + \tfrac{1}{2}\rho V_2^2$$

In addition, the continuity equation (Eq. 3.16) can be written as

$$Q = A_1 V_1 = A_2 V_2$$

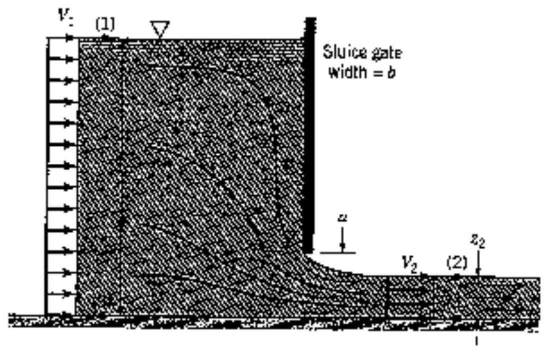
where A_2 is the small ($A_2 < A_1$) flow area at section (2). Combination of these two equations results in the following theoretical flowrate

$$Q = A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}} \tag{3.17}$$

The actual measured flowrate, Q_{actual}, will be smaller than this theoretical result because of various differences between the "real world" and the assumptions used in the derivation of Eq. 3.17. These differences (which are quite consistent and may be as small as 1 to 2% or as large as 40% depending on the geometry used) are discussed in Chapter 8.

EXAMPLE 3.11

Kerosene ($SG = 0.85$) flows through the Venturi meter shown in Fig. E3.11 with flowrates between 0.005 and 0.050 m^3/s. Determine the range in pressure difference, $p_1 - p_2$, needed to measure these flowrates.



■ FIGURE 3.12 Sluice gate geometry.

We apply the Bernoulli and continuity equations between points on the free surfaces at (1) and (2) to give

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

and

$$Q = A_1 V_1 = b V_1 z_1 = A_2 V_2 = b V_2 z_2$$

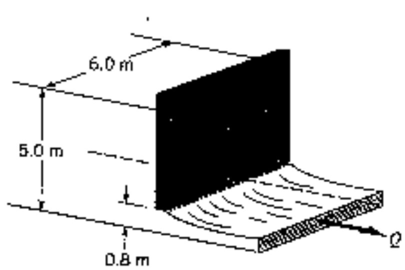
With the fact that $p_1 = p_2 = 0$, these equations can be combined to give the flowrate as

$$Q = z_2 b \sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}} \qquad (3.18)$$

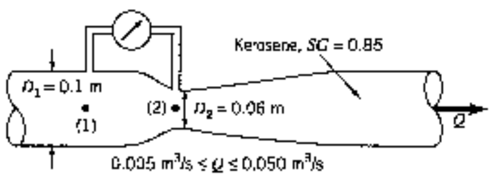
The downstream depth, z_2, not the gate opening, a, was used to obtain the result of Eq. 3.18 since a vena contracta results with a contraction coefficient, $C_c = z_2/a$, less than 1. Typically C_c is approximately 0.61 over the depth ratio range of $0 < a/z_1 < 0.2$. For larger values of a/z_1 the value of C_c increases rapidly.

EXAMPLE 3.12

Water flows under the sluice gate shown in Fig. E3.12. Determine the approximate flowrate per unit width of the channel.



■ FIGURE E3.12



■ FIGURE E3.11

SOLUTION

If the flow is assumed to be steady, inviscid, and incompressible, the relationship between flowrate and pressure is given by Eq. 3.17. This can be rearranged to give

$$p_1 - p_2 = \frac{Q^2\rho[1 - (A_2/A_1)^2]}{2A_2^2}$$

With a density of the flowing fluid of

$$\rho = SG\,\rho_{H_2O} = 0.85(1000\ \text{kg/m}^3) = 850\ \text{kg/m}^3$$

the pressure difference for the smallest flowrate is

$$p_1 - p_2 = (0.005\ \text{m}^3/\text{s})^2(850\ \text{kg/m}^3)\frac{[1 - (0.06\ \text{m}/0.10\ \text{m})^4]}{2[(\pi/4)(0.06\ \text{m})^2]^2}$$
$$= 1160\ \text{N/m}^2 = 1.16\ \text{kPa}$$

Likewise, the pressure difference for the largest flowrate is

$$p_1 - p_2 = (0.05)^2(850)\frac{[1 - (0.06/0.10)^4]}{2[(\pi/4)(0.06)^2]^2}$$
$$= 1.16\ 10^5\ \text{N/m}^2$$
$$= 116\ \text{kPa}$$

Thus,

$$1.16\ \text{kPa} \le p_1 - p_2 \le 116\ \text{kPa} \qquad \textbf{(Ans)}$$

These values represent the pressure differences for inviscid, steady, incompressible conditions. The ideal results presented here are independent of the particular flowmeter geometry—an orifice, nozzle, or Venturi meter (see Fig. 3.11).

It is seen from Eq. 3.17 that the flowrate varies as the square root of the pressure difference. Hence, as indicated by the numerical results, a tenfold increase in flowrate requires a one-hundredfold increase in pressure difference. This nonlinear relationship can cause difficulties when measuring flowrates over a wide range of values. Such measurements would require pressure transducers with a wide range of operation. An alternative is to use two flowmeters in parallel—one for the larger and one for the smaller flowrate ranges.

Other flowmeters based on the Bernoulli equation are used to measure flowrates in open channels such as flumes and irrigation ditches. The *sluice gate* as shown in Fig. 3.12 is an example.

SOLUTION

Under the assumptions of steady, inviscid, incompressible flow, we can apply Eq. 3.18 to obtain Q/b, the flowrate per unit width, as

$$\frac{Q}{b} = z_2\sqrt{\frac{2g(z_1 - z_2)}{1 - (z_2/z_1)^2}}$$

In this instance $z_1 = 5.0$ m and $a = 0.80$ m so the ratio $a/z_1 = 0.16 < 0.20$, and we can assume that the contraction coefficient is approximately $C_c = 0.61$. Thus, $z_2 = C_c a = 0.61$ (0.80 m) = 0.488 m and we obtain the flowrate

$$\frac{Q}{b} = (0.488\text{ m})\sqrt{\frac{2(9.81\text{ m/s}^2)(5.0\text{ m} - 0.488\text{ m})}{1 - (0.488\text{ m}/5.0\text{ m})^2}}$$
$$= 4.61\text{ m}^2/\text{s} \qquad \text{(Ans)}$$

If we consider $z_1 \gg z_2$ and neglect the kinetic energy of the upstream fluid, we would have

$$\frac{Q}{b} = z_2\sqrt{2gz_1} = 0.488\text{ m}\sqrt{2(9.81\text{ m/s}^2)(5.0\text{ m})} = 4.83\text{ m}^2/\text{s}$$

In this case the difference in Q with or without including V_1 is not too significant because the depth ratio is fairly large ($z_1/z_2 = 5.0/0.488 = 10.2$). Thus, it is often reasonable to neglect the kinetic energy upstream from the gate compared to that downstream of it.

3.7 The Energy Line and the Hydraulic Grade Line

A useful interpretation of the Bernoulli equation can be obtained through the use of the concepts of the *hydraulic grade line* (HGL) and the *energy line* (EL). These ideas represent a geometrical interpretation of a flow.

For steady, inviscid, incompressible flow the Bernoulli equation states that the sum of the pressure head, the velocity head, and the elevation head is constant along a streamline. This constant is called the *total head, H*.

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant on a streamline} = H \qquad (3.19)$$

The energy line is a line that represents the total head available to the fluid. As shown in Fig. 3.13, the elevation of the energy line can be obtained by measuring the stagnation pressure with a Pitot tube. The stagnation point at the end of the Pitot tube provides a measurement of the total head (or energy) of the flow. The static pressure tap connected to the piezometer tube shown, on the other hand, measures the sum of the pressure head and the elevation head, $p/\gamma + z$. This sum is often called the *piezometric head*.

A Pitot tube at another location in the flow will measure the same total head, as is shown in the figure. The elevation head, velocity head, and pressure head may vary along the streamline, however.

the fluid "not too viscous," and the flowrate "not too large," the above result may be accurate. If any of these assumptions are relaxed, a more detailed analysis is required (see Chapter 8). If the end of the hose were closed so the flowrate were zero, the hydraulic grade line would coincide with the energy line ($V^2/2g = 0$ throughout), the pressure at (1) would be greater than atmospheric, and water would leak through the hole at (1).

3.8 Restrictions on the Use of the Bernoulli Equation

One of the main assumptions in deriving the Bernoulli equation is that the fluid is incompressible. Although this is reasonable for most liquid flows, it can, in certain instances, introduce considerable errors for gases.

In the previous section we saw that the stagnation pressure is greater than the static pressure by an amount $\rho V^2/2$, provided that the density remains constant. If this dynamic pressure is not too large compared with the static pressure, the density change between points is not very large and the flow can be considered incompressible. However, since the dynamic pressure varies as V^2, the error associated with the assumption that a fluid is incompressible increases with the square of the velocity of the fluid.

A "rule of thumb" is that the flow of a perfect gas may be considered as incompressible provided the Mach number is less than about 0.3. The Mach number, $\text{Ma} = V/c$, is the ratio of the fluid speed, V, to the speed of sound in the fluid, c. In standard air ($T_1 =$ ... $c_1 = \sqrt{kRT_1} = 1117$ ft/s) this corresponds to a speed of $V_1 = c_1\text{Ma}_1 = 0.3\,(1117\text{ ft/s}) =$ ... ft/s $= 228$ mi/hr. At higher speeds, compressibility may become important.

Another restriction of the Bernoulli equation (Eq. 3.6) is the assumption that the flow is steady. For such flows, on a given streamline the velocity is a function of only s, the location along the streamline. That is, along a streamline $V = V(s)$. For unsteady flows the velocity is also a function of time, so that along a streamline $V = V(s,t)$. Thus, when taking the time derivative of the velocity to obtain the streamwise acceleration, we obtain $a_s = \partial V/\partial t + V\partial V/\partial s$ rather than just $a_s = V\partial V/\partial s$ as is true for steady flow. The unsteady term, $\partial V/\partial t$, does not allow the equation of motion to be easily integrated (as was done to obtain the Bernoulli equation) unless additional assumptions are made.

Another restriction on the Bernoulli equation is that the flow is inviscid. Recall that the Bernoulli equation is actually a first integral of Newton's second law along a streamline. This general integration was possible because, in the absence of viscous effects, the system considered was a conservative system. The total energy of the system remains constant. If viscous effects are important, the system is nonconservative and energy losses occur. A more detailed analysis is needed for these cases. Such material is presented in Chapter 8.

The final basic restriction on use of the Bernoulli equation is that there are no mechanical devices (pumps or turbines) in the system between the two points along the streamline for which the equation is applied. These devices represent sources or sinks of energy. Since the Bernoulli equation is actually one form of the energy equation, it must be altered to include pumps or turbines, if these are present. The inclusion of pumps and turbines is covered in Chapter 5.