93. $\int \sqrt{3x^2+5}\,dx = \frac{1}{\sqrt{3}}\int \sqrt{3x^2+5}\cdot\sqrt{3}\,dx = \frac{1}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}x\sqrt{3x^2+5} + \frac{5}{2}\ln(\sqrt{3}\,x+\sqrt{3x^2+5})\right] + C$

$= \frac{1}{2}x\sqrt{3x^2+5} + \frac{5\sqrt{3}}{6}\ln(\sqrt{3}\,x+\sqrt{3x^2+5}) + C$

94. $\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\,\text{arc sen}\,\frac{x+1}{2} + C$

95. $\int \sqrt{4x^2-4x+5}\,dx = \frac{1}{2}\int \sqrt{(2x-1)^2+4}\cdot 2\,dx$

$= \frac{1}{2}\left[\frac{2x-1}{2}\sqrt{4x^2-4x+5} + 2\ln(2x-1+\sqrt{4x^2-4x+5})\right] + C$

$= \frac{2x-1}{4}\sqrt{4x^2-4x+5} + \ln(2x-1+\sqrt{4x^2-4x+5}) + C$

Problemas propuestos

Comprobar las siguientes integraciones.

96. $\int (4x^3+3x^2+2x+5)\,dx = x^4+x^3+x^2+5x+C$

97. $\int (3-2x-x^4)\,dx = 3x - x^2 - x^5/5 + C$

98. $\int (2-3x+x^3)\,dx = 2x - 3x^2/2 + x^4/4 + C$

99. $\int (x^2-1)^2\,dx = x^5/5 - 2x^3/3 + x + C$

100. $\int (\sqrt{x} - \frac{1}{2}x + 2/\sqrt{x})\,dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4x^{1/2} + C$

101. $\int (a+x)^3\,dx = \frac{1}{4}(a+x)^4 + C$

102. $\int (x-2)^{3/2}\,dx = \frac{2}{5}(x-2)^{5/2} + C$

103. $\int \frac{dx}{x^3} = -\frac{1}{2x^2} + C$

104. $\int \frac{dx}{(x-1)^3} = -\frac{1}{2(x-1)^2} + C$

105. $\int \frac{dx}{\sqrt{x+3}} = 2\sqrt{x+3} + C$

106. $\int \sqrt{3x-1}\,dx = \frac{2}{9}(3x-1)^{3/2} + C$

107. $\int \sqrt{2-3x}\,dx = -\frac{2}{9}(2-3x)^{3/2} + C$

108. $\int (2x^2+3)^{1/3}x\,dx = \frac{3}{16}(2x^2+3)^{4/3} + C$

109. $\int (x-1)^2x\,dx = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + C$

110. $\int (x^2-1)x\,dx = \frac{1}{4}(x^2-1)^2 + C$

111. $\int \sqrt{1+y^4}\,y^3\,dy = \frac{1}{6}(1+y^4)^{3/2} + C$

112. $\int (x^3+3)x^2\,dx = \frac{1}{6}(x^3+3)^2 + C$

113. $\int (4-x^2)^2x^2\,dx = \frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{1}{7}x^7 + C$

114. $\int \frac{dy}{(2-y)^3} = \frac{1}{2(2-y)^2} + C$

115. $\int \frac{x\,dx}{(x^2+4)^3} = -\frac{1}{4(x^2+4)^2} + C$

116. $\int (1-x^3)^2\,dx = x - \frac{1}{2}x^4 + \frac{1}{7}x^7 + C$

117. $\int (1-x^3)^2x\,dx = \frac{1}{2}x^2 - \frac{2}{5}x^5 + \frac{1}{8}x^8 + C$

118. $\int (1-x^3)^2x^2\,dx = -\frac{1}{9}(1-x^3)^3 + C$

119. $\int (x^2-x)^4(2x-1)\,dx = \frac{1}{5}(x^2-x)^5 + C$

120. $\int \frac{3t\,dt}{\sqrt[3]{t^2+3}} = \frac{9}{4}(t^2+3)^{2/3} + C$

121. $\int \frac{(x+1)\,dx}{\sqrt{x^2+2x-4}} = \sqrt{x^2+2x-4} + C$

122. $\int \frac{dx}{(a+bx)^{1/3}} = \frac{3}{2b}(a+bx)^{2/3} + C$

123. $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx = \frac{2}{3}(1+\sqrt{x})^3 + C$

124. $\int \sqrt{x}\,(3-5x)\,dx = 2x^{3/2}(1-x) + C$

125. $\int \frac{(x+1)(x-2)}{\sqrt{x}}\,dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} - 4x^{1/2} + C$

126. $\int \frac{dx}{x-1} = \ln|x-1| + C$

127. $\int \frac{dx}{3x+1} = \frac{1}{3}\ln|3x+1| + C$

128. $\int \frac{3x\,dx}{x^2+2} = \frac{3}{2}\ln(x^2+2) + C$

129. $\int \frac{x^2\,dx}{1-x^3} = -\frac{1}{3}\ln|1-x^3| + C$

130. $\int \frac{x-1}{x+1}\,dx = x - 2\ln|x+1| + C$

131. $\int \frac{x^2+2x+2}{x+2}\,dx = \frac{1}{2}x^2 + 2\ln|x+2| + C$

132. $\int \frac{x+1}{x^2+2x+2}\,dx = \frac{1}{2}\ln(x^2+2x+2) + C$

133. $\int\left(\frac{dx}{2x-1} - \frac{dx}{2x+1}\right) = \ln\sqrt{\left|\frac{2x-1}{2x+1}\right|} + C$

134. $\int a^{4x}\,dx = \frac{1}{4}\frac{a^{4x}}{\ln a} + C$

135. $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$

136. $\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}e^{1/x^2} + C$

137. $\int e^{-x^2+2}x\,dx = -\frac{1}{2}e^{-x^2+2} + C$

138. $\int x^2e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$

139. $\int (e^x+1)^2\,dx = \frac{1}{2}e^{2x} + 2e^x + x + C$

140. $\int (e^x - x^e)\,dx = e^x - \frac{x^{e+1}}{e+1} + C$

141. $\int (e^x+1)^2e^x\,dx = \frac{1}{3}(e^x+1)^3 + C$

142. $\int \frac{e^{2x}}{e^{2x}+3}\,dx = \frac{1}{2}\ln(e^{2x}+3) + C$

143. $\int \left(e^x + \frac{1}{e^x}\right)^2 dx = \frac{1}{2}e^{2x} + 2x - \frac{1}{2e^{2x}} + C$

144. $\int \frac{e^x-1}{e^x+1}\,dx = \ln(e^x+1)^2 - x + C$

145. $\int \frac{e^{2x}-1}{e^{2x}+3}\,dx = \ln(e^{2x}+3)^{2/3} - \frac{1}{3}x + C$

146. $\int \frac{dx}{\sqrt{x}(1-\sqrt{x})} = \ln\frac{C}{(1-\sqrt{x})^2},\quad C>0$

147. $\int \frac{dx}{x+x^{1/3}} = \frac{3}{2}\ln C(x^{2/3}+1),\quad C>0$

148. $\int \operatorname{sen} 2x\,dx = -\frac{1}{2}\cos 2x + C$

149. $\int \cos\frac{1}{2}x\,dx = 2\operatorname{sen}\frac{1}{2}x + C$

150. $\int \sec 3x \operatorname{tag} 3x\,dx = \frac{1}{3}\operatorname{tag} 3x + C$

151. $\int \csc^2 2x\,dx = -\frac{1}{2}\cot 2x + C$

152. $\int x\sec^2 x^2\,dx = \frac{1}{2}\operatorname{tag} x^2 + C$

153. $\int \operatorname{tag}^2 x\,dx = \operatorname{tag} x - x + C$

154. $\int \operatorname{tag}\frac{1}{2}x\,dx = 2\ln|\sec\frac{1}{2}x| + C$

155. $\int \csc 3x\,dx = \frac{1}{3}\ln|\csc 3x - \cot 3x| + C$

156. $\int b\sec ax \operatorname{tag} ax\,dx = \frac{b}{a}\sec ax + C$

157. $\int (\cos x - \operatorname{sen} x)^2\,dx = x + \frac{1}{2}\cos 2x + C$

158. $\int \operatorname{sen} ax\cos ax\,dx = \frac{1}{2a}\operatorname{sen}^2 ax + C$

$= -\frac{1}{2a}\cos^2 ax + C' = -\frac{1}{4a}\cos 2ax + K$

159. $\int \operatorname{sen}^3 x\cos x\,dx = \frac{1}{4}\operatorname{sen}^4 x + C$

160. $\int \cos^4 x\operatorname{sen} x\,dx = -\frac{1}{5}\cos^5 x + C$

161. $\int \operatorname{tag}^5 x\sec^2 x\,dx = \frac{1}{6}\operatorname{tag}^6 x + C$

162. $\int \cot^4 3x\csc^2 3x\,dx = -\frac{1}{15}\cot^5 3x + C$

163. $\int \frac{dx}{1-\operatorname{sen}\frac{1}{2}x} = 2(\operatorname{tag}\frac{1}{2}x + \sec\frac{1}{2}x) + C$

164. $\int \frac{dx}{1+\cos 3x} = \frac{1-\cos 3x}{3\operatorname{sen} 3x} + C$

165. $\int \frac{dx}{1+\sec ax} = x + \frac{1}{a}(\cot ax - \csc ax) + C$

166. $\int \sec^2\frac{x}{a}\operatorname{tag}\frac{x}{a}\,dx = \frac{1}{2}a\operatorname{tag}^2\frac{x}{a} + C$

167. $\int \frac{\sec^2 3x}{\operatorname{tag} 3x}\,dx = \frac{1}{3}\ln|\operatorname{tag} 3x| + C$

168. $\int \frac{\sec^5 x}{\csc x}\,dx = \frac{1}{4}\sec^4 x + C$

169. $\int e^{\operatorname{tag} 2x} \sec^2 2x \, dx = \frac{1}{2} e^{\operatorname{tag} 2x} + C$

170. $\int e^{2 \operatorname{sen} 3x} \cos 3x \, dx = \frac{1}{6} e^{2 \operatorname{sen} 3x} + C$

171. $\int \frac{dx}{\sqrt{5 - x^2}} = \operatorname{arc\,sen} \frac{x\sqrt{5}}{5} + C$

172. $\int \frac{dx}{5 + x^2} = \frac{\sqrt{5}}{5} \operatorname{arc\,tag} \frac{x\sqrt{5}}{5} + C$

173. $\int \frac{dx}{x\sqrt{x^2 - 5}} = \frac{\sqrt{5}}{5} \operatorname{arc\,sec} \frac{x\sqrt{5}}{5} + C$

174. $\int \frac{e^x \, dx}{\sqrt{1 - e^{2x}}} = \operatorname{arc\,sen} e^x + C$

175. $\int \frac{e^{2x} \, dx}{1 + e^{4x}} = \frac{1}{2} \operatorname{arc\,tag} e^{2x} + C$

176. $\int \frac{dx}{\sqrt{4 - 9x^2}} = \frac{1}{3} \operatorname{arc\,sen} \frac{3x}{2} + C$

177. $\int \frac{dx}{9x^2 + 4} = \frac{1}{6} \operatorname{arc\,tag} \frac{3x}{2} + C$

178. $\int \frac{\operatorname{sen} 8x}{9 + \operatorname{sen}^2 4x} dx = \frac{1}{12} \operatorname{arc\,tag} \frac{\operatorname{sen}^2 4x}{3} + C$

179. $\int \frac{\sec^2 x \, dx}{\sqrt{1 - 4 \operatorname{tag}^2 x}} = \frac{1}{2} \operatorname{arc\,sen} (2 \operatorname{tag} x) + C$

180. $\int \frac{dx}{x\sqrt{4 - 9 \ln^2 x}} = \frac{1}{3} \operatorname{arc\,sen} \ln x^{3/2} + C$

181. $\int \frac{2x^4 - x^2}{2x^2 + 1} dx = \frac{1}{3} x^3 - x + \frac{\sqrt{2}}{2} \operatorname{arc\,tag} x\sqrt{2} + C$

182. $\int \frac{\cos 2x \, dx}{\operatorname{sen}^2 2x + 8} = \frac{\sqrt{2}}{8} \operatorname{arc\,tag} \frac{\operatorname{sen} 2x}{2\sqrt{2}} + C$

183. $\int \frac{(2x - 3) \, dx}{x^2 + 6x + 13} = \int \frac{(2x + 6) \, dx}{x^2 + 6x + 13} - 9 \int \frac{dx}{x^2 + 6x + 13} = \ln (x^2 + 6x + 13) - \frac{9}{2} \operatorname{arc\,tag} \frac{x + 3}{2} + C$

184. $\int \frac{(x - 1) \, dx}{3x^2 - 4x + 3} = \frac{1}{6} \int \frac{(6x - 4) \, dx}{3x^2 - 4x + 3} - \int \frac{dx}{9x^2 - 12x + 9} = \frac{1}{6} \ln (3x^2 - 4x + 3) - \frac{\sqrt{5}}{15} \operatorname{arc\,tag} \frac{3x - 2}{\sqrt{5}} + C$

185. $\int \frac{x \, dx}{\sqrt{27 + 6x - x^2}} = -\sqrt{27 + 6x - x^2} + 3 \operatorname{arc\,sen} \frac{x - 3}{6} + C$

186. $\int \frac{(5 - 4x) \, dx}{\sqrt{12x - 4x^2 - 8}} = \sqrt{12x - 4x^2 - 8} - \frac{1}{2} \operatorname{arc\,sen} (2x - 3) + C$

187. $\int \frac{dx}{x^2 - 4} = \frac{1}{4} \ln \left| \frac{x - 2}{x + 2} \right| + C$

188. $\int \frac{dx}{4x^2 - 9} = \frac{1}{12} \ln \left| \frac{2x - 3}{2x + 3} \right| + C$

189. $\int \frac{dx}{9 - x^2} = \frac{1}{6} \ln \left| \frac{x + 3}{x - 3} \right| + C$

190. $\int \frac{dx}{25 - 9x^2} = \frac{1}{30} \ln \left| \frac{3x + 5}{3x - 5} \right| + C$

191. $\int \frac{dx}{\sqrt{x^2 + 4}} = \ln (x + \sqrt{x^2 + 4}) + C$

192. $\int \frac{dx}{\sqrt{4x^2 - 25}} = \frac{1}{2} \ln | 2x + \sqrt{4x^2 - 25} | + C$

193. $\int \sqrt{16 - 9x^2} \, dx = \frac{1}{2} x\sqrt{16 - 9x^2} + \frac{8}{3} \operatorname{arc\,sen} \frac{3x}{4} + C$

194. $\int \sqrt{x^2 - 16} \, dx = \frac{1}{2} x\sqrt{x^2 - 16} - 8 \ln | x + \sqrt{x^2 - 16} | + C$

195. $\int \sqrt{4x^2 + 9} \, dx = \frac{1}{2} x\sqrt{4x^2 + 9} + \frac{9}{4} \ln (2x + \sqrt{4x^2 + 9}) + C$

196. $\int \sqrt{x^2 - 2x - 3} \, dx = \frac{1}{2}(x - 1) \sqrt{x^2 - 2x - 3} - 2 \ln | x - 1 + \sqrt{x^2 - 2x - 3} | + C$

197. $\int \sqrt{12 + 4x - x^2} \, dx = \frac{1}{2}(x - 2) \sqrt{12 + 4x - x^2} + 8 \operatorname{arc\,sen} \frac{1}{4}(x - 2) + C$

198. $\int \sqrt{x^2 + 4x} \, dx = \frac{1}{2}(x + 2) \sqrt{x^2 + 4x} - 2 \ln | x + 2 + \sqrt{x^2 + 4x} | + C$

199. $\int \sqrt{x^2 - 8x} \, dx = \frac{1}{2}(x - 4) \sqrt{x^2 - 8x} - 8 \ln | x - 4 + \sqrt{x^2 - 8x} | + C$

200. $\int \sqrt{6x - x^2} \, dx = \frac{1}{2} (x - 3) \sqrt{6x - x^2} + \frac{9}{2} \operatorname{arc\,sen} \frac{x - 3}{3} + C$

Capítulo 26

Integración por partes

INTEGRACION POR PARTES. Sean u y v funciones derivables de x. En estas condiciones,

$$d(uv) = u\,dv + v\,du$$
$$u\,dv = d(uv) - v\,du$$

(i) $$\int u\,dv = uv - \int v\,du$$

Para aplicar (i) en la práctica, se separa el integrando en dos partes; una de ellas se iguala a u y la otra, junto con dx, a dv. (Por esta razón, este método se denomina *integración por partes.*) Es conveniente tener en cuenta los dos criterios siguientes:

(*a*) La parte que se iguala a dv debe ser fácilmente integrable.

(*b*) $\int v\,du$ no debe ser más complicada que $\int u\,dv$.

Ejemplo 1: Calcular $\int x^3 e^{x^2}\,dx$.

Hacemos $u = x^2$ y $dv = e^{x^2} x\,dx$; de donde $du = 2x\,dx$ y $v = \frac{1}{2}e^{x^2}$. Aplicando la fórmula

$$\int x^3 e^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \int x\,e^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \tfrac{1}{2}e^{x^2} + C$$

Ejemplo 2: Calcular $\int \ln(x^2+2)\,dx$.

Hacemos $u = \ln(x^2+2)$ y $dv = dx$; de donde $du = \dfrac{2x\,dx}{x^2+2}$ y $v = x$. Por tanto,

$$\int \ln(x^2+2)\,dx = x\ln(x^2+2) - \int \frac{2x^2\,dx}{x^2+2} = x\ln(x^2+2) - \int\left(2 - \frac{4}{x^2+2}\right)dx$$
$$= x\ln(x^2+2) - 2x + 2\sqrt{2}\,\text{arc tag}\,x/\sqrt{2} + C$$

(Ver Problemas 1-10.)

FORMULAS DE REDUCCION. Las *fórmulas de reducción* permiten simplificar el cálculo cuando se haya de aplicar la integración por partes varias veces consecutivas. (Ver Problema 9.) En general, una fórmula de reducción es aquella que da lugar a una nueva integral de la misma forma que la original, pero con un exponente mayor o menor. Una fórmula de reducción es útil si, finalmente, conduce a una integral que se pueda calcular fácilmente. Algunas de las fórmulas más corrientes de reducción son:

(A) $$\int \frac{du}{(a^2 \pm u^2)^m} = \frac{1}{a^2}\left\{\frac{u}{(2m-2)(a^2 \pm u^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{du}{(a^2 \pm u^2)^{m-1}}\right\}, \quad m \neq 1$$

(B) $$\int (a^2 \pm u^2)^m\,du = \frac{u(a^2 \pm u^2)^m}{2m+1} + \frac{2ma^2}{2m+1}\int (a^2 \pm u^2)^{m-1}\,du, \quad m \neq -1/2$$

(C) $$\int \frac{du}{(u^2 - a^2)^m} = -\frac{1}{a^2}\left\{\frac{u}{(2m-2)(u^2 - a^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{du}{(u^2 - a^2)^{m-1}}\right\}, \quad m \neq 1$$

(D) $$\int (u^2 - a^2)^m\,du = \frac{u(u^2 - a^2)^m}{2m+1} - \frac{2ma^2}{2m+1}\int (u^2 - a^2)^{m-1}\,du, \quad m \neq -1/2$$

(E) $$\int u^m e^{au}\,du = \frac{1}{a}u^m e^{au} - \frac{m}{a}\int u^{m-1} e^{au}\,du$$

(F) $$\int \operatorname{sen}^m u\,du = -\frac{\operatorname{sen}^{m-1} u \cos u}{m} + \frac{m-1}{m}\int \operatorname{sen}^{m-2} u\,du$$

(G) $$\int \cos^m u\,du = \frac{\cos^{m-1} u \operatorname{sen} u}{m} + \frac{m-1}{m}\int \cos^{m-2} u\,du$$

(H) $$\int \operatorname{sen}^m u \cos^n u\,du = \frac{\operatorname{sen}^{m+1} u \cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \operatorname{sen}^m u \cos^{n-2} u\,du$$

$$= -\frac{\operatorname{sen}^{m-1} u \cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \operatorname{sen}^{m-2} u \cos^n u\,du, \quad m \neq -n$$

(I) $$\int u^m \operatorname{sen} bu\,du = -\frac{u^m}{b}\cos bu + \frac{m}{b}\int u^{m-1}\cos bu\,du$$

(J) $$\int u^m \cos bu\,du = \frac{u^m}{b}\operatorname{sen} bu - \frac{m}{b}\int u^{m-1}\operatorname{sen} bu\,du$$

(Ver Problema 11.)

Problemas resueltos

1. Calcular $\int x \operatorname{sen} x\,dx$.

Podemos seguir los siguientes caminos:

(*a*) $u = x \operatorname{sen} x,\ dv = dx$; (*b*) $u = \operatorname{sen} x,\ dv = x\,dx$; (*c*) $u = x,\ dv = \operatorname{sen} x\,dx$.

(*a*) $u = x \operatorname{sen} x,\ dv = dx$. Por tanto $du = (\operatorname{sen} x + x\cos x)\,dx,\ v = x$, y

$$\int x \operatorname{sen} x\,dx = x \cdot x \operatorname{sen} x - \int x(\operatorname{sen} x + x\cos x)\,dx$$

La integral que resulta es menos sencilla que la original por la cual se descarta este camino.

(*b*) $u = \operatorname{sen} x,\ dv = x\,dx$. Por tanto $du = \cos x\,dx,\ v = \frac{1}{2}x^2$, y

$$\int x \operatorname{sen} x\,dx = \tfrac{1}{2}x^2 \operatorname{sen} x - \int \tfrac{1}{2}x^2 \cos x\,dx$$

La integral que resulta es menos sencilla que la original y también descartamos este camino.

(*c*) $u = x,\ dv = \operatorname{sen} x\,dx$. Por tanto $du = dx,\ v = -\cos x$, y

$$\int x \operatorname{sen} x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \operatorname{sen} x + C$$

2. Calcular $\int xe^x\,dx$.

Sea $u = x,\ dv = e^x\,dx$. Entonces, $du = dx,\ v = e^x$, y

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

3. Calcular $\int x^2 \ln x\,dx$.

Sea $u = \ln x,\ dv = x^2\,dx$. Por tanto, $du = \frac{dx}{x},\ v = \frac{x^3}{3}$, y

$$\int x^2 \ln x\,dx \quad \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{dx}{x} = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

4. Calcular $\int x\sqrt{1+x}\,dx$.

Haciendo $u = x,\ dv = \sqrt{1+x}\,dx$. Tendremos $du = dx,\ v = \frac{2}{3}(1+x)^{3/2}$, y

$$\int x\sqrt{1+x}\,dx = \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\int (1+x)^{3/2}\,dx = \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + C$$

5. Calcular $\int \text{arc sen } x\, dx$.

Haciendo $u = \text{arc sen } x$, $dv = dx$. Tendremos $du = dx/\sqrt{1-x^2}$, $v = x$, y

$$\int \text{arc sen } x\, dx = x \text{ arc sen } x - \int \frac{x\, dx}{\sqrt{1-x^2}} = x \text{ arc sen } x + \sqrt{1-x^2} + C$$

6. Calcular $\int \text{sen}^2 x\, dx$.

Haciendo $u = \text{sen } x$, $dv = \text{sen } x\, dx$. Tendremos $du = \cos x\, dx$, $v = -\cos x$, y

$$\int \text{sen}^2 x\, dx = -\text{sen } x \cos x + \int \cos^2 x\, dx$$

$$= -\text{sen } x \cos x + \int (1 - \text{sen}^2 x)\, dx = -\tfrac{1}{2} \text{sen } 2x + \int dx - \int \text{sen}^2 x\, dx$$

Pasando al primer miembro la integral del segundo,

$$2\int \text{sen}^2 x\, dx = -\tfrac{1}{2} \text{sen } 2x + x + C' \quad \text{y} \quad \int \text{sen}^2 x\, dx = \tfrac{1}{2}x - \tfrac{1}{4} \text{sen } 2x + C$$

7. Calcular $\int \sec^3 x\, dx$.

Haciendo $u = \sec x$, $dv = \sec^2 x\, dx$. Tendremos $du = \sec x \text{ tag } x\, dx$, $v = \text{tag } x$, y

$$\int \sec^3 x\, dx = \sec x \text{ tag } x - \int \sec x \text{ tag}^2 x\, dx = \sec x \text{ tag } x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \text{ tag } x - \int \sec^3 x\, dx + \int \sec x\, dx$$

Por tanto $2\int \sec^3 x\, dx = \sec x \text{ tag } x + \int \sec x\, dx = \sec x \text{ tag } x + \ln|\sec x + \text{tag } x| + C'$

y $\int \sec^3 x\, dx = \tfrac{1}{2}\{\sec x \text{ tag } x + \ln|\sec x + \text{tag } x|\} + C$

8. Calcular $\int x^2 \text{sen } x\, dx$.

Haciendo $u = x^2$, $dv = \text{sen } x\, dx$. Tendremos $du = 2x\, dx$, $v = -\cos x$, y

$$\int x^2 \text{sen } x\, dx = -x^2 \cos x + 2\int x \cos x\, dx$$

Haciendo en la integral resultante $u = x$ y $dv = \cos x\, dx$. Tendremos $du = dx$, $v = \text{sen } x$, y

$$\int x^2 \text{sen } x\, dx = -x^2 \cos x + 2\left\{x \text{ sen } x - \int \text{sen } x\, dx\right\}$$

$$= -x^2 \cos x + 2x \text{ sen } x + 2\cos x + C$$

9. Calcular $\int x^3 e^{2x}\, dx$.

Haciendo $u = x^3$, $dv = e^{2x}\, dx$. Tendremos $du = 3x^2\, dx$, $v = \tfrac{1}{2}e^{2x}$, y

$$\int x^3 e^{2x}\, dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 e^{2x}\, dx$$

Haciendo en la integral resultante $u = x^2$ y $dv = e^{2x}\, dx$. Tendremos $du = 2x\, dx$, $v = \tfrac{1}{2}e^{2x}$, y

$$\int x^3 e^{2x}\, dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\left\{\frac{1}{2}x^2 e^{2x} - \int x e^{2x}\, dx\right\} = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\int x e^{2x}\, dx$$

Haciendo en la integral resultante $u = x$ y $dv = e^{2x}\, dx$. Tendremos $du = dx$, $v = \tfrac{1}{2}e^{2x}$, y

$$\int x^3 e^{2x}\, dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\left\{\frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x}\, dx\right\}$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + C$$

10. (*a*) Haciendo $= x$, $dv = \dfrac{x\, dx}{(a^2 \pm x^2)^m}$; Tendremos $du = dx$, $v = \dfrac{\mp 1}{(2m-2)(a^2 \pm x^2)^{m-1}}$, y

$$\int \frac{x^2\, dx}{(a^2 \pm x^2)^m} = \frac{\mp x}{(2m-2)(a^2 \pm x^2)^{m-1}} \pm \frac{1}{2m-2}\int \frac{dx}{(a^2 \pm x^2)^{m-1}}$$

(*b*) Haciendo $u = x$, $dv = x(a^2 \pm x^2)^{m-1}\, dx$; Tendremos $du = dx$, $v = \dfrac{\pm 1}{2m}(a^2 \pm x^2)^m$, y

$$\int x^2 (a^2 \pm x^2)^{m-1}\, dx = \frac{\pm x}{2m}(a^2 \pm x^2)^m \mp \frac{1}{2m}\int (a^2 \pm x^2)^m\, dx$$

11. Hallar: (*a*) $\int \frac{dx}{(1+x^2)^{5/2}}$, (*b*) $\int (9+x^2)^{3/2}\,dx$.

(*a*) Como la fórmula de reducción (A) reduce a una unidad el exponente del denominador, aplicándola dos veces resulta:

$$\int \frac{dx}{(1+x^2)^{5/2}} = \frac{x}{3(1+x^2)^{3/2}} + \frac{2}{3}\int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{3(1+x^2)^{3/2}} + \frac{2}{3}\frac{x}{(1+x^2)^{1/2}} + C$$

(*b*) Aplicando la fórmula de reducción (B),

$$\int (9+x^2)^{3/2}\,dx = \frac{1}{4}x(9+x^2)^{3/2} + \frac{27}{4}\int (9+x^2)^{1/2}\,dx$$

$$= \frac{1}{4}x(9+x^2)^{3/2} + \frac{27}{8}\{x(9+x^2)^{1/2} + 9\ln(x+\sqrt{9+x^2})\} + C$$

Problemas propuestos

12. $\int x\cos x\,dx = x\,\text{sen}\,x + \cos x + C$

13. $\int x\sec^2 3x\,dx = \frac{x}{3}\,\text{tag}\,3x - \frac{1}{9}\ln|\sec 3x| + C$

14. $\int \text{arc cos}\,2x\,dx = x\,\text{arc cos}\,2x - \frac{1}{2}\sqrt{1-4x^2} + C$

15. $\int \text{arc tag}\,x\,dx = x\,\text{arc tag}\,x - \ln\sqrt{1+x^2} + C$

16. $\int x^2\sqrt{1-x}\,dx = -\frac{2}{105}(1-x)^{3/2}(15x^2+12x+8) + C$

17. $\int \frac{xe^x\,dx}{(1+x)^2} = \frac{e^x}{1+x} + C$

18. $\int x\,\text{arc tag}\,x\,dx = \frac{1}{2}(x^2+1)\,\text{arc tag}\,x - \frac{1}{2}x + C$

19. $\int x^2 e^{-3x}\,dx = -\frac{1}{3}e^{-3x}(x^2+\frac{2}{3}x+\frac{2}{9}) + C$

20. $\int \text{sen}^3 x\,dx = -\frac{2}{3}\cos^3 x - \text{sen}^2 x\cos x + C$

21. $\int x^3\,\text{sen}\,x\,dx = -x^3\cos x + 3x^2\,\text{sen}\,x + 6x\cos x - 6\,\text{sen}\,x + C$

22. $\int \frac{x\,dx}{\sqrt{a+bx}} = \frac{2(bx-2a)\sqrt{a+bx}}{3b^2} + C$

23. $\int \frac{x^2\,dx}{\sqrt{1+x}} = \frac{2}{15}(3x^2-4x+8)\sqrt{1+x} + C$

24. $\int x\,\text{arc sen}\,x^2\,dx = \frac{1}{2}x^2\,\text{arc sen}\,x^2 + \frac{1}{2}\sqrt{1-x^4} + C$

25. $\int \text{sen}\,x\,\text{sen}\,3x\,dx = \frac{1}{8}\,\text{sen}\,3x\cos x - \frac{3}{8}\,\text{sen}\,x\cos 3x + C$

26. $\int \text{sen}(\ln x)\,dx = \frac{1}{2}x(\text{sen}\ln x - \cos\ln x) + C$

27. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(b\,\text{sen}\,bx + a\cos bx)}{a^2+b^2} + C$

28. $\int e^{ax}\,\text{sen}\,bx\,dx = \frac{e^{ax}(a\,\text{sen}\,bx - b\cos bx)}{a^2+b^2} + C$

29. (*a*) Poniendo $\int \frac{a^2\,dx}{(a^2 \pm x^2)^m} = \int \frac{(a^2\pm x^2)\mp x^2}{(a^2\pm x^2)^m}dx = \int \frac{dx}{(a^2\pm x^2)^{m-1}} \mp \int \frac{x^2\,dx}{(a^2\pm x^2)^m}$ y aplicando el resultado del Problema 10 (*a*) deducir la fórmula de reducción (A).

(*b*) Poniendo $\int (a^2\pm x^2)^m\,dx = a^2\int (a^2\pm x^2)^{m-1}\,dx \pm \int x^2(a^2\pm x^2)^{m-1}\,dx$ y aplicando el resultado del Problema 10 (*b*) deducir la fórmula de reducción (B).

30. Deducir las fórmulas de reducción (C)-(J).

31. $\displaystyle\int \frac{dx}{(1-x^2)^3} = \frac{x(5-3x^2)}{8(1-x^2)^2} + \frac{3}{16}\ln\left|\frac{1+x}{1-x}\right| + C$

32. $\displaystyle\int \frac{dx}{(4+x^2)^{3/2}} = \frac{x}{4(4+x^2)^{1/2}} + C$

33. $\displaystyle\int (4-x^2)^{3/2}\,dx = \tfrac{1}{4}x(10-x^2)\sqrt{4-x^2} + 6 \text{ arc sen } \tfrac{1}{2}x + C$

34. $\displaystyle\int \frac{dx}{(x^2-16)^3} = \frac{1}{2048}\left\{\frac{x(3x^2-80)}{(x^2-16)^2} + \frac{3}{8}\ln\left|\frac{x-4}{x+4}\right|\right\} + C$

35. $\displaystyle\int (x^2-1)^{5/2}\,dx = \tfrac{1}{48}x(8x^4-26x^2+33)\sqrt{x^2-1} - \tfrac{5}{16}\ln|x+\sqrt{x^2-1}| + C$

36. $\displaystyle\int \text{sen}^4 x\,dx = \tfrac{3}{8}x - \tfrac{3}{8}\text{ sen } x\cos x - \tfrac{1}{4}\text{ sen}^3 x\cos x + C$

37. $\displaystyle\int \cos^5 x\,dx = \tfrac{1}{15}(3\cos^4 x + 4\cos^2 x + 8)\text{ sen } x + C$

38. $\displaystyle\int \text{sen}^3 x\cos^2 x\,dx = -\tfrac{1}{5}\cos^3 x\,(\text{sen}^2 x + \tfrac{2}{3}) + C$

39. $\displaystyle\int \text{sen}^4 x\cos^5 x\,dx = \tfrac{1}{9}\text{ sen}^5 x\,(\cos^4 x + \tfrac{4}{7}\cos^2 x + \tfrac{8}{35}) + C$

Otro procedimiento útil en los casos más complejos y laboriosos de esta sección, resulta al considerar que en (ver Problema 9)

(i) $$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + C$$

los términos del segundo miembro, sin tener en cuenta los coeficientes, se obtienen al derivar sucesivamente el integrando x^3e^{2x}. Así pues,

(ii) $$\int x^3e^{2x}\,dx = Ax^3e^{2x} + Bx^2e^{2x} + Dxe^{2x} + Ee^{2x} + C$$

y derivando

$$x^3e^{2x} = 2Ax^3e^{2x} + (3A+2B)x^2e^{2x} + (2B+2D)xe^{2x} + (D+2E)e^{2x}$$

Identificando coeficientes, tendremos

$$2A = 1,\quad 3A+2B = 0,\quad 2B+2D = 0,\quad D+2E = 0$$

De donde $A = \tfrac{1}{2}$, $B = -\tfrac{3}{2}A = -\tfrac{3}{4}$, $D = -B = \tfrac{3}{4}$, $E = -\tfrac{1}{2}D = -\tfrac{3}{8}$. Sustituyendo A, B, D, E en (ii), obtenemos (i).

Este método se puede aplicar en el cálculo de $\int f(x)\,dx$ siempre que al derivar repetidamente $f(x)$ se obtenga un número finito de términos diferentes.

40. Calcular $\int e^{2x}\cos 3x\,dx = {}^{1}/_{13}e^{2x}(3\text{ sen } 3x + 2\cos 3x) + C$ haciendo

$$\int e^{2x}\cos 3x\,dx = Ae^{2x}\text{ sen } 3x + Be^{2x}\cos 3x + C$$

41. Calcular $\int e^{3x}(2\text{ sen } 4x - 5\cos 4x)\,dx = {}^{1}/_{25}e^{3x}(-14\text{ sen } 4x - 23\cos 4x) + C$ haciendo

$$\int e^{3x}(2\text{ sen } 4x - 5\cos 4x)\,dx = Ae^{3x}\text{ sen } 4x + Be^{3x}\cos 4x + C$$

42. Calcular $\int \text{sen } 3x\cos 2x\,dx = -{}^{1}/_{5}(2\text{ sen } 3x\text{ sen } 2x + 3\cos 3x\cos 2x) + C$ haciendo

$$\int \text{sen } 3x\cos 2x\,dx = A\text{ sen } 3x\text{ sen } 2x + B\cos 3x\cos 2x + D\cos 3x\text{ sen } 2x + E\text{ sen } 3x\cos 2x + C$$

43. Calcular $\displaystyle\int e^{3x}x^2\text{ sen } x\,dx = \frac{e^{3x}}{250}[25x^2(3\text{ sen } x - \cos x) - 10x(4\text{ sen } x - 3\cos x) + 9\text{ sen } x - 13\cos x] + C$

Capítulo 27

Integrales trigonométricas

LAS IDENTIDADES que se utilizan en la resolución de las integrales trigonométricas de este capítulo son las siguientes:

1. $\operatorname{sen}^2 x + \cos^2 x = 1$
2. $1 + \operatorname{tag}^2 x = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$
4. $\operatorname{sen}^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\operatorname{sen} x \cos x = \frac{1}{2}\operatorname{sen} 2x$
7. $\operatorname{sen} x \cos y = \frac{1}{2}[\operatorname{sen}(x - y) + \operatorname{sen}(x + y)]$
8. $\operatorname{sen} x \operatorname{sen} y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$
9. $\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$
10. $1 - \cos x = 2\operatorname{sen}^2 \frac{1}{2}x$
11. $1 + \cos x = 2\cos^2 \frac{1}{2}x$
12. $1 \pm \operatorname{sen} x = 1 \pm \cos(\frac{1}{2}\pi - x)$

Problemas resueltos

SENOS Y COSENOS

1. $\int \operatorname{sen}^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\operatorname{sen} 2x + C$

2. $\int \cos^2 3x\, dx = \int \frac{1}{2}(1 + \cos 6x)\, dx = \frac{1}{2}x + \frac{1}{12}\operatorname{sen} 6x + C$

3. $\int \operatorname{sen}^3 x\, dx = \int \operatorname{sen}^2 x \operatorname{sen} x\, dx = \int (1 - \cos^2 x)\operatorname{sen} x\, dx = -\cos x + \frac{1}{3}\cos^3 x + C$

4. $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \operatorname{sen}^2 x)^2 \cos x\, dx$

$$= \int \cos x\, dx - 2\int \operatorname{sen}^2 x \cos x\, dx + \int \operatorname{sen}^4 x \cos x\, dx$$

$$= \operatorname{sen} x - \frac{2}{3}\operatorname{sen}^3 x + \frac{1}{5}\operatorname{sen}^5 x + C$$

5. $\int \operatorname{sen}^2 x \cos^3 x\, dx = \int \operatorname{sen}^2 x \cos^2 x \cos x\, dx = \int \operatorname{sen}^2 x (1 - \operatorname{sen}^2 x)\cos x\, dx$

$$= \int \operatorname{sen}^2 x \cos x\, dx - \int \operatorname{sen}^4 x \cos x\, dx = \frac{1}{3}\operatorname{sen}^3 x - \frac{1}{5}\operatorname{sen}^5 x + C$$

6. $\int \cos^4 2x \operatorname{sen}^3 2x\, dx = \int \cos^4 2x \operatorname{sen}^2 2x \operatorname{sen} 2x\, dx = \int \cos^4 2x (1 - \cos^2 2x)\operatorname{sen} 2x\, dx$

$$= \int \cos^4 2x \operatorname{sen} 2x\, dx - \int \cos^6 2x \operatorname{sen} 2x\, dx = -\frac{1}{10}\cos^5 2x + \frac{1}{14}\cos^7 2x + C$$

7. $\int \operatorname{sen}^3 3x \cos^5 3x\, dx = \int (1 - \cos^2 3x)\cos^5 3x \operatorname{sen} 3x\, dx$

$$= \int \cos^5 3x \operatorname{sen} 3x\, dx - \int \cos^7 3x \operatorname{sen} 3x\, dx = -\frac{1}{18}\cos^6 3x + \frac{1}{24}\cos^8 3x + C$$

o

$$\int \operatorname{sen}^3 3x \cos^5 3x\, dx = \int \operatorname{sen}^3 3x (1 - \operatorname{sen}^2 3x)^2 \cos 3x\, dx$$

$$= \int \operatorname{sen}^3 3x \cos 3x\, dx - 2\int \operatorname{sen}^5 3x \cos 3x\, dx + \int \operatorname{sen}^7 3x \cos 3x\, dx$$

$$= \frac{1}{12}\operatorname{sen}^4 3x - \frac{1}{9}\operatorname{sen}^6 3x + \frac{1}{24}\operatorname{sen}^8 3x + C$$

8. $\int \cos^3 \frac{x}{3}\,dx = \int \left(1 - \operatorname{sen}^2 \frac{x}{3}\right) \cos \frac{x}{3}\,dx = 3 \operatorname{sen} \frac{x}{3} - \operatorname{sen}^3 \frac{x}{3} + C$

9. $\int \operatorname{sen}^4 x\,dx = \int (\operatorname{sen}^2 x)^2\,dx = \frac{1}{4}\int (1 - \cos 2x)^2\,dx$

$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x\,dx + \frac{1}{4}\int \cos^2 2x\,dx$

$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x\,dx + \frac{1}{8}\int (1 + \cos 4x)\,dx$

$= \frac{1}{4}x - \frac{1}{4}\operatorname{sen} 2x + \frac{1}{8}x + \frac{1}{32}\operatorname{sen} 4x + C = \frac{3}{8}x - \frac{1}{4}\operatorname{sen} 2x + \frac{1}{32}\operatorname{sen} 4x + C$

10. $\int \operatorname{sen}^2 x \cos^2 x\,dx = \frac{1}{4}\int \operatorname{sen}^2 2x\,dx = \frac{1}{8}\int (1 - \cos 4x)\,dx = \frac{1}{8}x - \frac{1}{32}\operatorname{sen} 4x + C$

11. $\int \operatorname{sen}^4 3x \cos^2 3x\,dx = \int (\operatorname{sen}^2 3x \cos^2 3x) \operatorname{sen}^2 3x\,dx = \frac{1}{8}\int \operatorname{sen}^2 6x\,(1 - \cos 6x)\,dx$

$= \frac{1}{8}\int \operatorname{sen}^2 6x\,dx - \frac{1}{8}\int \operatorname{sen}^2 6x \cos 6x\,dx$

$= \frac{1}{16}\int (1 - \cos 12x)\,dx - \frac{1}{8}\int \operatorname{sen}^2 6x \cos 6x\,dx$

$= \frac{1}{16}x - \frac{1}{192}\operatorname{sen} 12x - \frac{1}{144}\operatorname{sen}^3 6x + C$

12. $\int \operatorname{sen} 3x \operatorname{sen} 2x\,dx = \int \frac{1}{2}\{\cos(3x - 2x) - \cos(3x + 2x)\}\,dx = \frac{1}{2}\int (\cos x - \cos 5x)\,dx$

$= \frac{1}{2}\operatorname{sen} x - \frac{1}{10}\operatorname{sen} 5x + C$

13. $\int \operatorname{sen} 3x \cos 5x\,dx = \int \frac{1}{2}\{\operatorname{sen}(3x - 5x) + \operatorname{sen}(3x + 5x)\}\,dx = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$

14. $\int \cos 4x \cos 2x\,dx = \frac{1}{2}\int (\cos 2x + \cos 6x)\,dx = \frac{1}{4}\operatorname{sen} 2x + \frac{1}{12}\operatorname{sen} 6x + C$

15. $\int \sqrt{1 - \cos x}\,dx = \sqrt{2}\int \operatorname{sen} \tfrac{1}{2}x\,dx = -2\sqrt{2}\cos \tfrac{1}{2}x + C$

16. $\int (1 + \cos 3x)^{3/2}\,dx = 2\sqrt{2}\int \cos^3 \tfrac{3}{2}x\,dx = 2\sqrt{2}\int (1 - \operatorname{sen}^2 \tfrac{3}{2}x) \cos \tfrac{3}{2}x\,dx$

$= 2\sqrt{2}\,(\tfrac{2}{3}\operatorname{sen} \tfrac{3}{2}x - \tfrac{2}{9}\operatorname{sen}^3 \tfrac{3}{2}x) + C$

17. $\int \frac{dx}{\sqrt{1 - \operatorname{sen} 2x}} = \int \frac{dx}{\sqrt{1 - \cos(\frac{1}{2}\pi - 2x)}} = \frac{\sqrt{2}}{2}\int \frac{dx}{\operatorname{sen}(\frac{1}{4}\pi - x)} = \frac{\sqrt{2}}{2}\int \csc(\tfrac{1}{4}\pi - x)\,dx$

$= -\frac{\sqrt{2}}{2}\ln|\csc(\tfrac{1}{4}\pi - x) - \cot(\tfrac{1}{4}\pi - x)| + C$

TANGENTES, SECANTES, COTANGENTES, COSECANTES

18. $\int \operatorname{tag}^4 x\,dx = \int \operatorname{tag}^2 x \operatorname{tag}^2 x\,dx = \int \operatorname{tag}^2 x (\sec^2 x - 1)\,dx = \int \operatorname{tag}^2 x \sec^2 x\,dx - \int \operatorname{tag}^2 x\,dx$

$= \int \operatorname{tag}^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx = \tfrac{1}{3}\operatorname{tag}^3 x - \operatorname{tag} x + x + C$

19. $\int \operatorname{tag}^5 x\,dx = \int \operatorname{tag}^3 x \operatorname{tag}^2 x\,dx = \int \operatorname{tag}^3 x (\sec^2 x - 1)\,dx$

$= \int \operatorname{tag}^3 x \sec^2 x\,dx - \int \operatorname{tag}^3 x\,dx = \int \operatorname{tag}^3 x \sec^2 x\,dx - \int \operatorname{tag} x (\sec^2 x - 1)\,dx$

$= \tfrac{1}{4}\operatorname{tag}^4 x - \tfrac{1}{2}\operatorname{tag}^2 x + \ln|\sec x| + C$

20. $\int \sec^4 2x\,dx = \int \sec^2 2x \sec^2 2x\,dx = \int \sec^2 2x\,(1 + \operatorname{tag}^2 2x)\,dx$

$= \int \sec^2 2x\,dx + \int \operatorname{tag}^2 2x \sec^2 2x\,dx = \tfrac{1}{2}\operatorname{tag} 2x + \tfrac{1}{6}\operatorname{tag}^3 2x + C$

21. $\int \tan^3 3x \sec^4 3x\,dx = \int \tan^3 3x\,(1+\tan^2 3x)\sec^2 3x\,dx$

$= \int \tan^3 3x \sec^2 3x\,dx + \int \tan^5 3x \sec^2 3x\,dx = \frac{1}{12}\tan^4 3x + \frac{1}{18}\tan^6 3x + C$

22. $\int \tan^2 x \sec^3 x\,dx = \int (\sec^2 x - 1)\sec^3 x\,dx = \int \sec^5 x\,dx - \int \sec^3 x\,dx$

$= \frac{1}{4}\sec^3 x \tan x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x| + C$, integrando por partes.

23. $\int \tan^3 2x \sec^3 2x\,dx = \int \tan^2 2x \sec^2 2x \cdot \sec 2x \tan 2x\,dx$

$= \int (\sec^2 2x - 1)\sec^2 2x \cdot \sec 2x \tan 2x\,dx$

$= \int \sec^4 2x \cdot \sec 2x \tan 2x\,dx - \int \sec^2 2x \cdot \sec 2x \tan 2x\,dx$

$= \frac{1}{10}\sec^5 2x - \frac{1}{6}\sec^3 2x + C$

24. $\int \cot^3 2x\,dx = \int \cot 2x\,(\csc^2 2x - 1)\,dx = -\frac{1}{4}\cot^2 2x + \frac{1}{2}\ln|\csc 2x| + C$

25. $\int \cot^4 3x\,dx = \int \cot^2 3x\,(\csc^2 3x - 1)\,dx = \int \cot^2 3x \csc^2 3x\,dx - \int \cot^2 3x\,dx$

$= \int \cot^2 3x \csc^2 3x\,dx - \int (\csc^2 3x - 1)\,dx = -\frac{1}{9}\cot^3 3x + \frac{1}{3}\cot 3x + x + C$

26. $\int \csc^6 x\,dx = \int \csc^2 x\,(1+\cot^2 x)^2\,dx$

$= \int \csc^2 x\,dx + 2\int \cot^2 x \csc^2 x\,dx + \int \cot^4 x \csc^2 x\,dx$

$= -\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C$

27. $\int \cot 3x \csc^4 3x\,dx = \int \cot 3x\,(1+\cot^2 3x)\csc^2 3x\,dx$

$= \int \cot 3x \csc^2 3x\,dx + \int \cot^3 3x \csc^2 3x\,dx = -\frac{1}{6}\cot^2 3x - \frac{1}{12}\cot^4 3x + C$

28. $\int \cot^3 x \csc^5 x\,dx = \int \cot^2 x \csc^4 x \cdot \csc x \cot x\,dx = \int (\csc^2 x - 1)\csc^4 x \cdot \csc x \cot x\,dx$

$= \int \csc^6 x \cdot \csc x \cot x\,dx - \int \csc^4 x \cdot \csc x \cot x\,dx$

$= -\frac{1}{7}\csc^7 x + \frac{1}{5}\csc^5 x + C$

Problemas propuestos

29. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

30. $\int \sin^3 2x\,dx = \frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x + C$

31. $\int \sin^4 2x\,dx = \frac{3}{8}x - \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + C$

32. $\int \cos^4 \frac{1}{2}x\,dx = \frac{3}{8}x + \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C$

33. $\int \sin^7 x\,dx = \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + C$

34. $\int \cos^6 \frac{1}{2}x \, dx = \frac{5}{16}x + \frac{1}{2}\operatorname{sen} x + \frac{3}{32}\operatorname{sen} 2x - \frac{1}{24}\operatorname{sen}^3 x + C$

35. $\int \operatorname{sen}^2 x \cos^5 x \, dx = \frac{1}{3}\operatorname{sen}^3 x - \frac{2}{5}\operatorname{sen}^5 x + \frac{1}{7}\operatorname{sen}^7 x + C$

36. $\int \operatorname{sen}^3 x \cos^2 x \, dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

37. $\int \operatorname{sen}^3 x \cos^3 x \, dx = \frac{1}{48}\cos^3 2x - \frac{1}{16}\cos 2x + C$

38. $\int \operatorname{sen}^4 x \cos^4 x \, dx = \frac{1}{128}(3x - \operatorname{sen} 4x + \frac{1}{8}\operatorname{sen} 8x) + C$

39. $\int \operatorname{sen} 2x \cos 4x \, dx = \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$

40. $\int \cos 3x \cos 2x \, dx = \frac{1}{2}\operatorname{sen} x + \frac{1}{10}\operatorname{sen} 5x + C$

41. $\int \operatorname{sen} 5x \operatorname{sen} x \, dx = \frac{1}{8}\operatorname{sen} 4x - \frac{1}{12}\operatorname{sen} 6x + C$

42. $\int \frac{\cos^3 x \, dx}{1 - \operatorname{sen} x} = \operatorname{sen} x + \frac{1}{2}\operatorname{sen}^2 x + C$

43. $\int \frac{\cos^{2/3} x}{\operatorname{sen}^{8/3} x} dx = -\frac{3}{5}\cot^{5/3} x + C$

44. $\int \frac{\cos^3 x}{\operatorname{sen}^4 x} dx = \csc x - \frac{1}{3}\csc^3 x + C$

45. $\int x(\cos^3 x^2 - \operatorname{sen}^3 x^2) \, dx = \frac{1}{12}(\operatorname{sen} x^2 + \cos x^2)(4 + \operatorname{sen} 2x^2) + C$

46. $\int \operatorname{tag}^3 x \, dx = \frac{1}{2}\operatorname{tag}^2 x + \ln|\cos x| + C$

47. $\int \operatorname{tag}^3 3x \sec 3x \, dx = \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

48. $\int \operatorname{tag}^{3/2} x \sec^4 x \, dx = \frac{2}{5}\operatorname{tag}^{5/2} x + \frac{2}{9}\operatorname{tag}^{9/2} x + C$

49. $\int \operatorname{tag}^4 x \sec^4 x \, dx = \frac{1}{7}\operatorname{tag}^7 x + \frac{1}{5}\operatorname{tag}^5 x + C$

50. $\int \cot^3 x \, dx = -\frac{1}{2}\cot^2 x - \ln|\operatorname{sen} x| + C$

51. $\int \cot^3 x \csc^4 x \, dx = -\frac{1}{4}\cot^4 x - \frac{1}{6}\cot^6 x + C$

52. $\int \cot^3 x \csc^3 x \, dx = -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C$

53. $\int \csc^4 2x \, dx = -\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$

54. $\int \left(\frac{\sec x}{\operatorname{tag} x}\right)^4 dx = -\frac{1}{3\operatorname{tag}^3 x} - \frac{1}{\operatorname{tag} x} + C$

55. $\int \frac{\cot^3 x}{\csc x} dx = -\operatorname{sen} x - \csc x + C$

56. $\int \operatorname{tag} x \sqrt{\sec x} \, dx = 2\sqrt{\sec x} + C$

57. Aplicar la integración por partes para deducir las fórmulas de reducción

(a) $\int \sec^m u \, du = \frac{1}{m-1}\sec^{m-2} u \operatorname{tag} u + \frac{m-2}{m-1}\int \sec^{m-2} u \, du$

(b) $\int \csc^m u \, du = -\frac{1}{m-1}\csc^{m-2} u \cot u + \frac{m-2}{m-1}\int \csc^{m-2} u \, du$

Aplicar las fórmulas de reducción por partes del Problema 57 para resolver los Problemas 58-60.

58. $\int \sec^3 x \, dx = \frac{1}{2}\sec x \operatorname{tag} x + \frac{1}{2}\ln|\sec x + \operatorname{tag} x| + C$

59. $\int \csc^5 x \, dx = -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x + \frac{3}{8}\ln|\csc x - \cot x| + C$

60. $\int \sec^6 x \, dx = \frac{1}{5}\sec^4 x \operatorname{tag} x + \frac{4}{15}\sec^2 x \operatorname{tag} x + \frac{8}{15}\operatorname{tag} x + C$

$= \frac{1}{5}\operatorname{tag}^5 x + \frac{2}{3}\operatorname{tag}^3 x + \operatorname{tag} x + C$